財務數學

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11.5 Stochastic Calculus for Jump Process

Itô-Doeblin Formula for One Jump Process

$$X^c(t) = X^c(0) + \int_0^t \Gamma(s)\,dW(s) + \int_0^t \Theta(s)\,ds \qquad (11.5.1)$$

$$dX^c(s) = \Gamma(s)\,dW(s) + \Theta(s)\,ds, \quad dX^c(s)\,dX^c(s) = \Gamma^2(s)\,ds.$$

Itô-Doeblin Formula for One Jump Process

Let $f(x)$ be a function whose first and second derivatives are defined and continuous. Then

Itô's lemma : $df(X(t)) = f'(X(t))dX(t) + \frac{1}{2}f''(X(t))(dX(t))^2$

$$
\begin{aligned}
df(X^c(s)) &= f'(X^c(s))\,dX^c(s) + \frac{1}{2}f''(X^c(s))\,dX^c(s)\,dX^c(s) \\
&= f'(X^c(s))\Gamma(s)\,dW(s) + f'(X^c(s))\Theta(s)\,ds \\
&\qquad + \frac{1}{2}f''(X^c(s))\Gamma^2(s)\,ds. \qquad (11.5.2)
\end{aligned}
$$

$$dX^c(s) = \Gamma(s)\,dW(s) + \Theta(s)\,ds, \quad dX^c(s)\,dX^c(s) = \Gamma^2(s)\,ds.$$

Itô-Doeblin Formula for One Jump Process

We write this in integral form as

$$\int_0^t df(X^c(s)) = f(X^c(s))\Big|_0^t = f(X^c(t)) - f(X^c(0))$$

$$= \int_0^t f'(X^c(s))\Gamma(s)\,dW(s) + \int_0^t f'(X^c(s))\theta(s)\,ds + \frac{1}{2}\int_0^t f''(X^c(s))\Gamma^2(s)\,ds$$

$$f\big(X^c(t)\big) = f\big(X^c(0)\big) + \int_0^t f'\big(X^c(s)\big)\Gamma(s)\,dW(s) + \int_0^t f'\big(X^c(s)\big)\Theta(s)\,ds$$
$$+\frac{1}{2}\int_0^t f''\big(X^c(s)\big)\Gamma^2(s)\,ds.$$

Itô-Doeblin Formula for One Jump Process

We now add a right-continuous pure jump term J into (11.5.1), setting

$$X(t) = X(0) + I(t) + R(t) + J(t),$$

Between jumps of J, the analogue of (11.5.2) holds:

$$dJ = 0, \quad dX(s) = \Gamma(s)dW(s) + \Theta(s)ds = dX^c(s), \; dX(s)dX(s) = \Gamma^2(s)ds$$

$$\begin{aligned} df\big(X(s)\big) &= f'\big(X(s)\big)\,dX(s) + \frac{1}{2}f''\big(X(s)\big)\,dX(s)\,dX(s) \\ &= f'\big(X(s)\big)\Gamma(s)\,dW(s) + f'\big(X(s)\big)\Theta(s)\,ds \\ &\quad + \frac{1}{2}f''\big(X(s)\big)\Gamma^2(s)\,ds \\ &= f'\big(X(s)\big)\,dX^c(s) + \frac{1}{2}f''\big(X(s)\big)\,dX^c(s)\,dX^c(s). \qquad (11.5.3) \end{aligned}$$

Theorem 11.5.1

Theorem 11.5.1 (Itô-Doeblin formula for one jump process). *Let $X(t)$ be a jump process and $f(x)$ a function for which $f'(x)$ and $f''(x)$ are defined and continuous.*

PROOF: Fix $\omega \in \Omega$, which fixes the path of X, and let $0 < \tau_1 < \tau_2 < \cdots < \tau_{n-1} < t$ be the jump times in $[0, t)$ of this path of the process X. We set $\tau_0 = 0$, which is not a jump time, and $\tau_n = t$, which may or may not be a jump time. Whenever $u < v$ are both in the same interval (τ_j, τ_{j+1}), there is no jump between times u and v, and the Itô-Doeblin formula (11.5.3) for continuous processes applies. We thus have

$$f(X(v)) - f(X(u)) = \int_u^v f'(X(s))\, dX^c(s) + \frac{1}{2} \int_u^v f''(X(s))\, dX^c(s)\, dX^c(s).$$

Theorem 11.5.1

Letting $u \downarrow \tau_j$ and $v \uparrow \tau_{j+1}$ and using the right-continuity of X, we conclude that

Before the Jump

After the Jump

$$f(X(\tau_{j+1}-)) - f(X(\tau_j))$$
$$= \int_{\tau_j}^{\tau_{j+1}} f'(X(s))\, dX^c(s) + \frac{1}{2}\int_{\tau_j}^{\tau_{j+1}} f''(X(s))\, dX^c(s)\, dX^c(s). \quad (11.5.5)$$

We now add the jump in $f(X)$ at time τ_{j+1} into (11.5.5), obtaining thereby

$$f(X(\tau_{j+1})) - f(X(\tau_j))$$
$$= \int_{\tau_j}^{\tau_{j+1}} f'(X(s))\, dX^c(s) + \frac{1}{2}\int_{\tau_j}^{\tau_{j+1}} f''(X(s))\, dX^c(s)\, dX^c(s)$$
$$+ f(X(\tau_{j+1})) - f(X(\tau_{j+1}-)).$$

$$\begin{cases} X(t) = X(0) + I(t) + R(t) + J(t) \\ X(t-) = X(0) + I(t) + R(t) + J(t-) \end{cases}$$

Theorem 11.5.1

Summing over $j = 0, \ldots, n-1$, we obtain

$$\begin{aligned} f\big(X(t)\big) - f\big(X(0)\big) \\ &= \sum_{j=0}^{n-1} \left[f\big(X(\tau_{j+1})\big) - f\big(X(\tau_j)\big) \right] \\ &= \int_0^t f'\big(X(s)\big)\, dX^c(s) + \frac{1}{2} \int_0^t f''\big(X(s)\big)\, dX^c(s)\, dX^c(s) \\ &\qquad + \sum_{j=0}^{n-1} \left[f\big(X(\tau_{j+1})\big) - f\big(X(\tau_{j+1}-)\big) \right], \end{aligned}$$

Note in this connection that if there is no jump at $\tau_n = t$, then the last term in the sum on the right-hand side, $f\big(X(\tau_n)\big) - f\big(X(\tau_n-)\big)$, is zero. □

Theorem 11.5.1

Theorem 11.5.1 (Itô-Doeblin formula for one jump process). *Let $X(t)$ be a jump process and $f(x)$ a function for which $f'(x)$ and $f''(x)$ are defined and continuous. Then*

$$f(X(t)) = f(X(0)) + \int_0^t f'(X(s))\, dX^c(s) + \frac{1}{2}\int_0^t f''(X(s))\, dX^c(s)\, dX^c(s) + \sum_{0<s\le t} \left[f(X(s)) - f(X(s-)) \right]. \tag{11.5.4}$$

Example 11.5.2

Example 11.5.2 (Geometric Poisson process). Consider the geometric Poisson process

$$S(t) = S(0)\exp\left\{N(t)\log(\sigma+1) - \lambda\sigma t\right\} = S(0)e^{-\lambda\sigma t}(\sigma+1)^{N(t)}, \quad (11.5.6)$$

where $\sigma > -1$ is a constant. If $\sigma > 0$, this process jumps up and moves down between jumps; if $-1 < \sigma < 0$, it jumps down and moves up between jumps.

We may write $S(t) = S(0)f\big(X(t)\big)$, where $f(x) = e^x$ and

$$X(t) = N(t)\log(\sigma+1) - \lambda\sigma t$$

has continuous part $X^c(t) = -\lambda\sigma t$ and pure jump part $J(t) = N(t)\log(\sigma+1)$.

Example 11.5.2

According to the Itô-Doeblin formula for jump processes,

$$S(t) = f\big(X(t)\big) = f\big(X(0)\big) - \lambda\sigma \int_0^t f'\big(X(u)\big)\,du + \sum_{0<u\le t} \big[f\big(X(u)\big) - f\big(X(u-)\big)\big]$$

Itô-Doeblin Formula

$$f\big(X(t)\big) = f\big(X(0)\big) + \int_0^t f'\big(X(s)\big)\,dX^c(s) + \frac{1}{2}\int_0^t f''\big(X(s)\big)\,dX^c(s)\,dX^c(s) + \sum_{0<s\le t} \big[f\big(X(s)\big) - f\big(X(s-)\big)\big].$$

$f(x) = e^x = f'(x)$

$f'(X(u)) = e^{X(u)}$

$X^c(u) = -\lambda\sigma u, \quad dX^c(u) = -\lambda\sigma\,du$

$dX^c(u)\,dX^c(u) = \lambda^2\sigma^2 \underbrace{(du)^2}_{0} = 0$

$f(X(t))$

$= f(X(0)) + \int_0^t f'(X(u)) \times (-\lambda\sigma\,du)$

$+ 0 + \sum_{0<u\le t} [f(X(u)) - f(X(u-))]$

$= f(X(0)) - \lambda\sigma \int_0^t f'(X(u))\,du$

$+ \sum_{0<u\le t} [f(X(u)) - f(X(u-))]$

Example 11.5.2

$$\begin{aligned}S(t) &= f\big(X(t)\big)\\ &= f\big(X(0)\big) - \lambda\sigma\int_0^t f'\big(X(u)\big)\,du + \sum_{0<u\le t}\big[f\big(X(u)\big) - f\big(X(u-)\big)\big]\\ &= S(0) - \lambda\sigma\int_0^t S(u)\,du + \sum_{0<u\le t}\big[S(u) - S(u-)\big].\end{aligned} \tag{11.5.7}$$

$S(u) = f(x) = e^{x} = f'(x)$

$f'(X(u)) = e^{X(u)} = S(u)$

Example 11.5.2

If there is a jump at time u, then $S(u) = (\sigma + 1)S(u-)$. Therefore,

$$S(u) - S(u-) = \sigma S(u-) \tag{11.5.8}$$

$$S(u) = S(0)e^{-\lambda\sigma u}(\sigma+1)^{N(u)}$$

$$S(u-) = S(0)e^{-\lambda\sigma(u-)}(\sigma+1)^{N(u-)}$$

$$N(u) - N(u-) = 1$$

$$\frac{S(u)}{S(u-)} = e^{-\lambda\sigma\underbrace{(u-u-)}_{\to 0}}(\sigma+1)$$

Example 11.5.2

$$\begin{cases} \textbf{If there is a jump at time } \boldsymbol{u}, \Delta N(u) = 1. \\ \textbf{If there is no jump at time } \boldsymbol{u}, \Delta N(u) = 0. \end{cases}$$

$$\Rightarrow S(u) - S(u-) = \sigma S(u-)\Delta N(u).$$

$$\sum_{0<u\leq t} [S(u) - S(u-)] = \sum_{0<u\leq t} \sigma S(u-)\Delta N(u) = \sigma \int_0^t S(u-)\, dN(u).$$

Example 11.5.2

It does not matter whether we write the Riemann integral on the right-hand side of (11.5.7) as $\int_0^t S(u)\,du$ or as $\int_0^t S(u-)\,du$. The integrands in these two integrals differ at only finitely many times, and when we integrate with respect to du, these differences do not matter. Therefore, we may rewrite (11.5.7) as

$$\begin{aligned} S(t) &= S(0) - \lambda\sigma \int_0^t S(u-)\,du + \sigma \int_0^t S(u-)\,dN(u) \\ &= S(0) + \sigma \int_0^t S(u-)\,dM(u), \end{aligned}$$

$$M(u) = N(u) - \lambda u, \; dM(u) = dN(u) - \lambda du$$

Example 11.5.2

$$S(t) = S(0) - \lambda\sigma \int_0^t S(u-)\,du + \sigma \int_0^t S(u-)\,dN(u)$$
$$= S(0) + \sigma \int_0^t S(u-)\,dM(u),$$

M is the compensated Poisson process $M(u) = N(u) - \lambda u$, which is a martingale. Because the integrand $S(u-)$ is left-continuous, Theorem 11.4.5 guarantees that $S(t)$ is a martingale.

Theorem 11.4.5. *Assume that the jump process $X(s)$ of (11.4.1)–(11.4.3) is a martingale, the integrand $\Phi(s)$ is* left-continuous *and adapted, and*

$$\mathbb{E} \int_0^t \Gamma^2(s)\Phi^2(s)\,ds < \infty \text{ for all } t \geq 0.$$

Then the stochastic integral $\int_0^t \Phi(s)\,dX(s)$ is also a martingale.

Example 11.5.2

In this case, the Itô-Doeblin formula (11.5.7) has a differential form, namely,

$$dS(t) = \sigma S(t-)\, dM(t) = -\lambda\sigma S(t)\, dt + \sigma S(t-)\, dN(t). \tag{11.5.9}$$

We were able to obtain this differential form because in (11.5.8) we were able to write the jump in $f(X)$ (i.e., the jump in S) at time u in terms of $f\big(X(u-)\big)$ (i.e., in terms of $S(u-)$). $\square$

Corollary 11.5.3

Corollary 11.5.3. *Let* $W(t)$ *be a Brownian motion and let* $N(t)$ *be a Poisson process with intensity* $\lambda > 0$*, both defined on the same probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *and relative to the same filtration* $\mathcal{F}(t)$, $t \geq 0$. *Then the processes* $W(t)$ *and* $N(t)$ *are independent.*

Corollary 11.5.3

Key Step in Proof: Let u_1 and u_2 be fixed real numbers and define

$$Y(t) = \exp\left\{u_1 W(t) + u_2 N(t) - \frac{1}{2}u_1^2 t - \lambda\left(e^{u_2} - 1\right)t\right\}.$$

To do this, we define

$$X(s) = u_1 W(s) + u_2 N(s) - \frac{1}{2}u_1^2 s - \lambda\left(e^{u_2} - 1\right)s$$

and $f(x) = e^x$, so that $Y(s) = f\left(X(s)\right)$. The process $X(s)$ has Itô integral part $I(s) = u_1 W(s)$, Riemann integral part $R(s) = -\frac{1}{2}u_1^2 s - \lambda\left(e^{u_2} - 1\right)s$, and pure jump part $J(s) = u_2 N(s)$.

$$dX^c(s) = u_1\, dW(s) - \frac{1}{2}u_1^2\, ds - \lambda\left(e^{u_2} - 1\right) ds, \quad dX^c(s)\, dX^c(s) = u_1^2\, ds.$$

Corollary 11.5.3

We next observe that if Y has a jump at time s, then

$$Y(s) = \exp\left\{u_1 W(s) + u_2(N(s-)+1) - \frac{1}{2}u_1^2 s - \lambda\big(e^{u_2}-1\big)s\right\} = Y(s-)e^{u_2}.$$

Therefore, $Y(s) - Y(s-) = \big(e^{u_2} - 1\big)Y(s-)\Delta N(s)$.

$$
\begin{aligned}
Y(t) &= f\big(X(t)\big) \\
&= f\big(X(0)\big) + \int_0^t f'\big(X(s)\big)\, dX^c(s) + \frac{1}{2}\int_0^t f''\big(X(s)\big)\, dX^c(s)\, dX^c(s) \\
&\qquad + \sum_{0<s\le t} \left[f\big(X(s)\big) - f\big(X(s-)\big)\right]
\end{aligned}
$$

$$
dX^c(s) = u_1\, dW(s) - \frac{1}{2}u_1^2\, ds - \lambda\big(e^{u_2} - 1\big)\, ds, \quad dX^c(s)\, dX^c(s) = u_1^2\, ds.
$$

$$
f(X(0)) = \exp\{u_1 W(0) + u_2 N(0) - \tfrac{1}{2}u_1^2 \times 0 - \lambda(e^{u_2} - 1)\times 0\} = \exp\{0\} = 1
$$

$$
f'(X(s)) = \exp\{X(s)\} = Y(s)
$$

$$
\begin{aligned}
&= 1 + u_1\int_0^t Y(s)\, dW(s) - \frac{1}{2}u_1^2\int_0^t Y(s)\, ds - \lambda\big(e^{u_2} - 1\big)\int_0^t Y(s)\, ds \\
&\qquad + \frac{1}{2}u_1^2\int_0^t Y(s)\, ds + \sum_{0<s\le t}\left[Y(s) - Y(s-)\right]
\end{aligned}
$$

Corollary 11.5.3

$$= 1 + u_1 \int_0^t Y(s)\,dW(s) - \frac{1}{2}u_1^2 \int_0^t Y(s)\,ds - \lambda(e^{u_2} - 1) \int_0^t Y(s)\,ds$$
$$+ \frac{1}{2}u_1^2 \int_0^t Y(s)\,ds + \sum_{0<s\le t} [Y(s) - Y(s-)]$$

because Y has only finitely many jumps, $\int_0^t Y(s)\,ds = \int_0^t Y(s-)\,ds$.
$Y(s) - Y(s-) = (e^{u_2} - 1)Y(s-)\Delta N(s)$.

$$= 1 + u_1 \int_0^t Y(s)\,dW(s) - \lambda(e^{u_2} - 1) \int_0^t Y(s-)\,ds$$
$$+ (e^{u_2} - 1) \int_0^t Y(s-)\,dN(s)$$

Corollary 11.5.3

$$= 1 + u_1 \int_0^t Y(s)\,dW(s) - \lambda\left(e^{u_2} - 1\right) \int_0^t Y(s-)\,ds$$
$$+\left(e^{u_2} - 1\right) \int_0^t Y(s-)\,dN(s)$$

$$\boldsymbol{M(u) = N(u) - \lambda u,\ dM(u) = dN(u) - \lambda u}$$

$$= 1 + u_1 \int_0^t Y(s)\,dW(s) + \left(e^{u_2} - 1\right) \int_0^t Y(s-)\,dM(s), \qquad (11.5.10)$$

Corollary 11.5.3

$$Y(t) = 1 + u_1 \int_0^t Y(s)\, dW(s) + \left(e^{u_2} - 1\right) \int_0^t Y(s-)\, dM(s), \qquad (11.5.10)$$

The Itô integral $\int_0^t Y(s)\, dW(s)$ in the last line of (11.5.10) is a martingale, and the integral of the left-continuous process $Y(s-)$ with respect to the martingale $M(s)$ is also. Therefore, Y is a martingale.

Corollary 11.5.3

Because $Y(0) = 1$ and Y is a martingale, we have $\mathbb{E}Y(t) = 1$ for all t. In other words,

$$\mathbb{E}\exp\left\{u_1 W(t) + u_2 N(t) - \frac{1}{2}u_1^2 t - \lambda\left(e^{u_2} - 1\right)t\right\} = 1 \text{ for all } t \geq 0.$$

We have obtained the joint moment-generating function formula

$$\mathbb{E}e^{u_1 W(t) + u_2 N(t)} = \exp\left\{\frac{1}{2}u_1^2 t\right\} \cdot \exp\left\{\lambda t\left(e^{u_2} - 1\right)\right\}.$$

Corollary 11.5.3

$$\mathbb{E}e^{u_1 W(t)+u_2 N(t)} = \exp\left\{\frac{1}{2}u_1^2 t\right\} \cdot \exp\left\{\lambda t\left(e^{u_2}-1\right)\right\}.$$

This is the product of the moment-generating function $\mathbb{E}e^{u_1 W(t)} = \exp\left\{\frac{1}{2}u_1^2 t\right\}$ for $W(t)$ (see Exercise 1.6(i)) and the moment-generating function $\mathbb{E}e^{u_2 N(t)} = \exp\left\{\lambda t\left(e^{u_2}-1\right)\right\}$ for $N(t)$ (see (11.3.4)). Since the joint moment-generating function factors into the product of moment-generating functions, the random variables $W(t)$ and $N(t)$ are independent.

Corollary 11.5.3

Exercise 1.6. Let u be a fixed number in $\mathbb{R}$, and define the convex function $\varphi(x) = e^{ux}$ for all $x \in \mathbb{R}$. Let X be a normal random variable with mean $\mu = \mathbb{E}X$ and standard deviation $\sigma = \left[\mathbb{E}(X-\mu)^2\right]^{\frac{1}{2}}$, i.e., with density

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}.$$

(i) Verify that

$$\mathbb{E}e^{uX} = e^{u\mu + \frac{1}{2}u^2\sigma^2}.$$

$W(0) = 0, W(t) \sim N(0,t)$, MGF of $W(t) = e^{\frac{1}{2}u^2 t}$

When $y = 1$, we have the Poisson process, whose moment-generating function is thus

$$\varphi_{N(t)}(u) = \mathbb{E}e^{uN(t)} = \exp\{\lambda t(e^u - 1)\}. \quad (11.3.4)$$